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Constant Rank Matrices

Harlan D. Mills

CONSTANT RANK MATRICES

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Harlan Duncan Mills

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Approved:

B. Vinograde
In Charge of Major Work

D. L. Holl
Head of Major Department

R M Hixon
Dean of Graduate College

Iowa State College

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I. INTRODUCTION

Statement of the Problem

A matrix is a rectangular array of quantities which, as an array, obeys certain rules when combined with other matrices by the operations of addition and multiplication, or when combined with scalar quantities by the operation of multiplication. These operations have meaning if and only if the matric quantities and scalar quantities are elements of a ring. A minor of a matrix is a certain function of a square sub-array of the matrix, and has a unique meaning if and only if the elements of the sub-array are commutative. The rank of a matrix is a function of all possible minors of the matrix, and thus a matrix will have a unique rank if and only if the matric quantities are elements of a commutative ring, say R.

If the matric quantities are variable elements of R, and R' is a sub-ring of R, we make

Definition I. A matrix is of constant rank r over R' if and only if it is of exactly rank r for every possible set of matric quantities in R'.

In particular, let the matric quantities be elements of a ring of polynomials of a scalar variable, t, with coefficients in a field F, say

$$A(t) \equiv (a_{ij}^{o} + a_{ij}^{1}t + \cdots + a_{ij}^{k}t^{k}) \equiv A_o + A_1 t + \cdots + A_k t^{k}$$

where a_{ij}^{k} is in F. Let F' be a sub-field of the algebraic closure of F, then restating Def. I, we have

Definition II. A(t) is of constant rank r over F' if and only if A(t) is of exactly rank r for every t in F'.

It is the purpose of this paper to discuss the properties of A(t) when it is of constant rank, and to develope certain criteria for the recognition of such matrices.

Discussion of Contents

The reader's acquaintance with elementary matrix theory is assumed, including, for Chapter II, elementary transformations, invariant factor theory, and for Chapter III, partitioned matrices, the characteristic function, and similarity transformations.

In Chapter II, the general matric polynomial is treated by means of invariant factor theory, and a generic form for a constant rank matric polynomial is given.

Chapter III is independent of Chapter II and deals with the linear matrix. A canonical form for a matrix is developed which displays the existence of a constant rank. A partial treatment of the multilinear matrix

$$A(t_1,t_2, \cdots,t_k) \equiv A_0 + A_1t_1 + A_2t_2 + \cdots + A_kt_k$$

follows in Chapter IV based on the results of Chapter III. Illustrations are given in an appendix.

Some Work of J. H. M. Wedderburn

J. H. M. Wedderburn[1] considers an integral set of vector polynomials defined in the following manner:

$$z_i(u) \equiv z_0^i u^{m_i} + z_1^i u^{m_i-1} + \cdots + z_{m_i}^i; \quad i = 1,2,\cdots,k; \quad (1)$$

is the integral set of vector polynomials where

u is a scalar variable,
z_j^i is a vector independent of u,
z_0^i is not a zero vector.

If $z_j^i \equiv \left[z_j^{i1}, z_j^{i2}, \cdots, z_j^{in}\right]$ is a column, then

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[1] Wedderburn, J. H. M. Lectures on matrices. Chap. IV, 1934. American Mathematical Society. Colloquium Publications, Volume XVII.

$$(z_1(u),\ z_2(u),\ \cdots,\ z_k(u)) \equiv z(u)$$

is a matric polynomial of k columns and n rows, and this is a convenient perspective for viewing these sets.

Wedderburn defines elementary transformations for the integral sets (1), and these correspond exactly to elementary column transformations on $z(u)$ in the integral domain of polynomials in u. He shows by means of these elementary transformations that a basis for the integral set can be obtained in which the leading coefficients of the basis set are linearly independent.

If the set

$$z_1(u),\ z_2(u),\ \cdots,\ z_k(u)$$

is linearly independent, that is, no c_i's exist such that

$$\sum_{i=1}^{k} c_i z_i(u) \equiv 0, \qquad c_i \neq 0, \text{ some } i,$$

there may exist a set of c_i's for a particular value of u, say u_1, so $p(u_1) = \sum_{i=1}^{k} c_i z_i(u_1) = 0$, and therefore, $(u - u_1)$ is a factor of $p(u)$. If u_1 is an α-fold zero of $p(u)$, $p'(u) = \frac{p(u)}{(u-u_1)^{\alpha}}$ is integral. Wedderburn uses this to derive from the integral set a new integral set, eventually arriving at an integral set,

$$x_1(u),\ x_2(u),\ \cdots,\ x_k(u),$$

for which no d_i's exist such that

$$\sum_{i=1}^{k} d_i x_i(u) = 0 \text{ for any } u.$$

He calls the set

$$x_1(u),\ x_2(u),\ \cdots,\ x_k(u),$$

an elementary set.

The correspondence to z(u) is the dividing out of the invariant factors of z(u), z(u) being reduced in the quotient field of polynomials in u. However, these elementary sets and their corresponding matric form, x(u), have the property that the rank is independent of the variable matric quantities.

The work cited here is the only reference to matrices whose ranks are independent of their variable matric quantities that the author has found in the literature.

II. MATRIC POLYNOMIALS OF CONSTANT RANK

Summary and Definitions

In this chapter, the matric polynomial,

$$A(t) \equiv A_0 + A_1 t + A_2 t^2 + \cdots + A_k t^k$$

will be considered where the matric quantities of A_i are elements of F, independent of t, and t is a scalar variable in F', a sub-field of the algebraic closure of F. The conditions for A(t) to be of constant rank will be formulated in terms of its invariant factors, and a generic form for A(t) will be given.

It will be assumed that F', the domain of the scalar variable, t, contains more than k(r+1) distinct elements, k being the highest power of t occurring in A(t) with a non-zero coefficient, and r being the rank of A(t).

A Necessary and Sufficient Condition for Constant Rank Matric Polynomials

An immediate consequence of Def. II is

Lemma I. A(t) is of constant rank r over F' if, and only if, every minor of order r + 1 is identically zero, and the greatest common divisor of the minors of order r has no zero in F'.

Proof. Assume A(t) is of constant rank r over F'. Every minor of order r + 1 is a polynomial in t of degree at most k(r+1). But this polynomial is zero for more than k(r+1) distinct elements of F', and hence, identically zero. For every value of t in F' there exists at least one non zero minor of order r. Thus the minors of order r have no common zero in F', and consequently the greatest common divisor of the minors of order r has no zero in F'.

Now assume every minor of order r + 1 is identically zero, and the greatest common divisor of minors of order r has no zero in F'. For every value of t in F' there exists a minor of order r not zero while every minor of order r + 1 is zero. Therefore, A(t) is of exactly rank r for every t in F', and the lemma is shown.

It is well known that under elementary transformations on a matrix, A(t), the greatest common divisors of minors of any order are unchanged, hence we have the following theorem.

Theorem I. The existence of constant rank, as well as the rank itself, is invariant under elementary transformations.

Proof. The rows and columns of every minor of order $r + 1$ are linearly dependent for every t in F', and will remain so under elementary transformations. Furthermore, the greatest common divisor of the minors of order r is unchanged by elementary transformations. Thus, by Lemma I the theorem is true.

There exist for $A(t)$, elementary transformations

$$P(t) \equiv P_0 + P_1 t + P_2 t^2 + \cdots + P_l t^l,$$

$$Q(t) \equiv Q_0 + Q_1 t + Q_2 t^2 + \cdots + Q_m t^m, \quad \text{with}$$

$$|P(t)| = K_1 \neq 0, \quad |Q(t)| = K_2 \neq 0, \quad K_1, K_2 \text{ in } F,$$

such that

$$P(t)A(t)Q(t) = \begin{pmatrix} \phi_1(t) & & & & & & \\ & \phi_2(t) & & & & & \\ & & \ddots & & & & \\ & & & \phi_r(t) & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{pmatrix}$$

$$= M(t),$$

where the $\phi_i(t)$ are the invariant factors of $A(t)$. So exhibited, $A(t)$ is of rank r, and is of constant rank r if and only if

$$\prod_{i=1}^{r} \phi_i(t) \neq 0. \qquad \text{for all } t \text{ in } F'.$$

We now prove the following theorem.

<u>Theorem II</u>. A necessary and sufficient condition for $A(t)$ to be of constant rank r over F' is that the first invariant factor $\phi_1(t)$ not vanish in F'.

Proof.

$$\phi_i(t) \mid \phi_j(t) \qquad \text{if } i \geq j,$$

and so $\phi_1(t)$ has every zero of

$$\phi_1(t), \phi_2(t), \cdots, \phi_r(t).$$

If $\phi_1(t) \neq 0$ for all t in F', then

$$\phi_2(t) \neq 0, \phi_3(t) \neq 0, \cdots, \phi_r(t) \neq 0$$

and

$$\prod_{i=1}^{r} \phi_i(t) \neq 0, \qquad \text{all } t \text{ in } F'.$$

But if $\phi_1(t_1) = 0$, t_1 in F', then $\prod_{i=1}^{r} \phi_i(t_1) = 0$, and the theorem is proved.

<u>Corollary</u>. If $F' = \overline{F}$, the algebraic closure of F, then $\phi_1(t) = 1$, for $\phi_1(t)$ is a polynomial in t and will have at least one zero in $\overline{F}$ if it is not constant.

A Generic Form

<u>Theorem III</u>. Let

$$N(t) \equiv \begin{pmatrix} \theta_1(t) & & & & & & \\ & \theta_2(t) & & & & & \\ & & \ddots & & & & \\ & & & \theta_r(t) & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{pmatrix}$$

where $\theta_i(t)$ is a monic polynomial in t,

$$\theta_i(t) \mid \theta_j(t) \qquad i \geq j,$$

$$\theta_1(t) \neq 0 \quad \text{for all } t \text{ in } F',$$

and $\prod_{i=1}^{r} \theta_i(t)$ is of degree at most kr, k being defined as on page 6; if

$$R(t) \equiv R_0 + R_1 t + \cdots + R_{l'} t^{l'}$$

$$S(t) \equiv S_0 + S_1 t + \cdots + S_{m'} t^{m'}$$

are elementary transformations, then $R(t)N(t)S(t)$ generates the entire class of constant rank matrices $A(t)$.

Proof.

$$A(t) = P^{-1}(t)P(t)A(t)Q(t)Q^{-1}(t)$$

$$= P^{-1}(t)M(t)Q^{-1}(t)$$

$$= R(t)N(t)S(t),$$

where $\phi_i \equiv \theta_i$, $P^{-1}(t) \equiv R(t)$, $Q^{-1}(t) \equiv S(t)$.

Note: If $N(t)$ is written

$$N(t) = N_o + N_1 t + N_2 t^2 + \dots + N_{kr} t^{kr}$$

and

$$A(t) \equiv R(t)N(t)S(t), \quad \text{then}$$

$$A_o + A_1 t + \dots + A_k t^k \equiv (R_o + R_1 t + \dots + R_{l'} t^{l'})(N_o + N_1 t + N_2 t^2 + \dots$$

$$+ N_{kr} t^{kr})(S_o + S_1 t + \dots + S_{m'} t^{m'}).$$

Equating like powers of t,

$$R_o N_o S_o = A_o,$$

$$R_1 N_o S_o + R_o N_o S_1 + R_o N_1 S_o = A_1,$$

$$\sum_{j=0}^{l} \sum_{i=0}^{j} F_i N_j S_{l-i-j} = A_l, \qquad l = 0,1,2,\cdots,k$$

$$= 0, \qquad l = k + 1, k + 4, \cdots .$$

Under these restrictions, $R(t)N(t)S(t)$ generates only the entire class of constant rank matrices, $A(t)$.

Corollary. If $A(t)$ is an elementary transformation, $N(t) = I$, where I is the identity matrix.

III. A CANONICAL FORM FOR A LINEAR CONSTANT RANK MATRIX

Summary and Definitions

In this chapter a canonical form for a linear matrix of constant rank r over F' will be developed. Consider $A(t) \equiv A_o + A_1 t$ as a member of the class of matric polynomials in Chapter II with k = 1, except that F' contains more than 2r + 1 distinct elements.

The Primary Canonical Form

Lemma II. If A(t) is of constant rank r over F', the rank of A_o is r.

Proof. 0 must be an element of F', and $A(0) = A_o$ must be exactly of rank r.

If A(t) is of constant rank, then there exist matrices independent of t, namely, P, Q, such that

$$P A_o Q = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \text{ where I is an r x r identity matrix}$$

and

$$P A(t) Q = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} + P A_1 Q t.$$

Partitioning PA_1Q arbitrarily into the same dimensions as PA_oQ

$$PA(t)Q = \begin{pmatrix} I + B_1t & B_2t \\ B_3t & B_4t \end{pmatrix}.$$

Consider a minor of order $r + 1$ consisting of the block $I + B_1t$ along with part of any one row of (B_3+B_4t) and any one column of $\begin{pmatrix} B_2t \\ B_4t \end{pmatrix}$. Obviously this minor is a polynomial in t having no constant term and of highest degree $r + 1$. Since the only constant terms of the minor occur along the diagonal, the coefficient of t^{r+1-k} must be a sum of products having k of its factors taken from the diagonal. But this polynomial is identically zero if A(t) is of constant rank r by Lemma I, or

$$a_1t + a_2t^2 + \cdots + a_{r+1}\, t^{r+1} = 0,\ a_i = 0, \qquad i = 1,2,\cdots,r+1.$$

<u>Lemma III</u>. A(t) is of constant rank r over F', only if $B_4 = 0$.

Proof: Consider the coefficient a_1 above. Now r of its elements must come from the diagonal of $(I + B_1t)$, and from I in particular. The only possible choice remaining is the element from the intersection of (B_3t, B_4t) and $\begin{pmatrix} B_2t \\ B_4t \end{pmatrix}$, an element of B_4. But if $a_1 = 0$, and an $r + 1$ order minor can be formed with every element of B_4, then every element of B_4 is zero and B_4 is a zero matrix. If $B_4 \neq 0$, then some minor of order $r + 1$ has $a_1 \neq 0$, and so there exists such a minor not identically zero.

With $B_4 = 0$, using the following notation for the $(r+1)$-rowed minor formed by bordering $I + B_1 t$,

$$\begin{vmatrix} 1+b^1_{11}t & b^1_{12}t & \cdots & b^1_{1r}t & b^2_{1j}t \\ b^1_{21}t & 1+b^1_{22}t & \cdots & b^1_{2r}t & b^2_{2j}t \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ b^1_{r1}t & b^1_{r2}t & \cdots & 1+b^1_{rr}t & b^2_{rj}t \\ b^3_{i1}t & b^3_{i2}t & \cdots & b^3_{ir}t & 0 \end{vmatrix},$$

we may prove the following lemma.

<u>Lemma IV</u>. If $B_4 = 0$, $A(t)$ is of constant rank r over F' only if

$$B_3 B_1^k B_2 = 0, \qquad k = 0,1,2,\cdots,r-1,$$

where

$$B_1^0 = I.$$

Proof. Consider a_2. If all but one of the 1's along the diagonal are used, the only possible terms are $- b^3_{ik} b^2_{kj}$, and summing,

$$a_s = -\sum_{k=1}^{r} b^s_{ik}\, b^s_{kj} = -(B_sB_s)_{ij},$$ where the symbol

$(B_sB_s)_{ij}$ means the element at the intersection of the i^{th} row and j^{th} column of the matrix B_sB_s. If $a_s = 0$, then $(B_sB_s)_{ij} = 0$ all i and j, hence $B_sB_s = 0$. If $B_sB_s \neq 0$, then some $(B_sB_s)_{ij} \neq 0$ and hence some minor of order $r + 1$ is not identically zero.

Now consider a_s. Writing a typical term of the sum,

$$\Delta_s = \begin{vmatrix} b^1_{xx} & b^1_{xy} & b^s_{xj} \\ b^1_{yx} & b^1_{yy} & b^s_{yj} \\ b^s_{ix} & b^3_{iy} & 0 \end{vmatrix}$$

and

$$a_s = \sum_{\substack{x=1 \\ y=x+1}}^{\substack{y=r \\ x=r-1}} \Delta_s.$$

If y and x were interchanged in Δ_s, one row interchange and one column interchange would be affected, hence the sign would remain the same, and

$$a_s = \sum_{\substack{y=1 \\ x=y+1}}^{\substack{x=r \\ y=r-1}} \Delta_s.$$

Further, if $x = y$, $\Delta_s = 0$ for the first two columns would be identical.

Thus,

$$2a_s = \sum_{\substack{x=1\\y=x+1}}^{\substack{y=r\\x=r-1}} \Delta_s + \sum_{\substack{y=1\\x=y+1}}^{\substack{x=r\\y=r-1}} \Delta_s + \sum_{x=y=1}^{x=y=r} \Delta_s = \sum_{\substack{x=1\\y=1}}^{\substack{y=r\\x=r}} \Delta_s.$$

But expanding by the last row

$$\sum_{\substack{x=1\\y=1}}^{\substack{y=r\\x=r}} \Delta_s = \sum_{\substack{x=1\\y=1}}^{\substack{y=r\\x=r}} (\overset{s}{b}_{ix} \overset{1}{b}_{xy} \overset{2}{b}_{yj} - \overset{s}{b}_{ix} \overset{1}{b}_{yy} \overset{2}{b}_{xj}$$

$$- \overset{s}{b}_{iy} \overset{1}{b}_{xx} \overset{2}{b}_{yj} + \overset{s}{b}_{iy} \overset{1}{b}_{yx} \overset{2}{b}_{xj})$$

$$= \sum_{\substack{x=1\\y=1}}^{\substack{y=r\\x=r}} \overset{s}{b}_{ix} \overset{1}{b}_{xy} \overset{2}{b}_{yj} + \overset{s}{b}_{iy} \overset{1}{b}_{yx} \overset{2}{b}_{xj}$$

$$- \sum_{y=1}^{r} \overset{1}{b}_{yy} \sum_{x=1}^{r} \overset{s}{b}_{ix} \overset{2}{b}_{xj} - \sum_{x=1}^{r} \overset{1}{b}_{xx} \sum_{y=1}^{r} \overset{s}{b}_{iy} \overset{2}{b}_{yj}.$$

Since the x and y are dummy indices over the same set of values, they may be interchanged, and,

$$2a_s = 2 \sum_{\substack{x=1\\y=1}}^{\substack{y=r\\x=r}} \overset{s}{b}_{ix} \overset{1}{b}_{xy} \overset{2}{b}_{yj} - 2 \sum_{y=1}^{r} \overset{1}{b}_{yy} \sum_{x=1}^{r} \overset{s}{b}_{ix} \overset{2}{b}_{xj}$$

$$= 2(B_s B_1 B_2)_{ij} - 2(\mathrm{Tr}\ B_1)(B_s B_2)_{ij},$$ where $(\mathrm{Tr}\ B_1)$ is the trace of B_1.

If $A(t)$ is of constant rank, $a_2 = 0$, $a_3 = 0$, and by consideration of a_2, $(B_3B_2)_{ij} = 0$. Hence $(B_3B_1B_2)_{ij} = 0$, all i and j, so

$$B_3B_1B_2 = 0.$$

If $(B_3B_1B_2) \neq 0$, some $(B_3B_1B_2)_{ij} \neq 0$ and if $a_2 = 0$, then $a_3 \neq 0$ and a minor of order $r + 1$ exists which is not identically zero.

The method of the proof is now clear. For a_{k+1}, consider the typical term,

$$\Delta_{k+1} = \begin{vmatrix} \overset{1}{b}_{x_1x_1} & \overset{1}{b}_{x_1x_2} & \cdots & \overset{1}{b}_{x_1x_k} & \overset{2}{b}_{x_1j} \\ \overset{1}{b}_{x_2x_1} & \overset{1}{b}_{x_2x_2} & \cdots & \overset{1}{b}_{x_2x_k} & \overset{2}{b}_{x_2j} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \overset{1}{b}_{x_kx_1} & \overset{1}{b}_{x_kx_2} & \cdots & \overset{1}{b}_{x_kx_k} & \overset{2}{b}_{x_kj} \\ \overset{3}{b}_{ix_1} & \overset{3}{b}_{ix_2} & \cdots & \overset{3}{b}_{ix_k} & 0 \end{vmatrix} .$$

This determinant will have $k!$ circulant products of the type

$$\overset{3}{b}_{ix_1} \overset{1}{b}_{x_1x_2} b_{x_2x_3} \cdots \overset{1}{b}_{x_{k-1}x_k} \overset{2}{b}_{x_kj}.$$

Furthermore, every other term can be partitioned into cycles so one cycle will be $b^{2}_{ix_\alpha}\, b^{1}_{x_\alpha x_\beta} \cdots b^{1}_{x_\pi x_\rho}\, b^{2}_{x_\rho j}$, and the remainder of the cycles will be only from B_1.

Now we may write

$$a_{k+1} = \sum_{t=1}^{k} \sum_{x_t=t}^{r-k+t} \Delta_{k+1}\,.$$

The interchange of x_i with x_j will interchange a row and column, leaving Δ_{k+1} unchanged. Furthermore, if $x_i = x_j$, then $\Delta_{k+1} = 0$, for two columns and rows will be identical. Therefore,

$$k!\, a_{k+1} = \sum_{t=1}^{k} \sum_{x_t=1}^{r} \Delta_{k+1}, \text{ hence}$$

$$\begin{aligned} k!\, a_{k+1} = \sum_{x_t=1}^{r} \Big[& k!(b^{2}_{ix_1}\, b^{1}_{x_1x_2} \cdots b^{1}_{x_{k-1}x_k}\, b^{2}_{x_kj})(-1)^{k+1} \\ & + f_1(b^{1}_{x_ix_j})(b^{2}_{ix_1}\, b^{1}_{x_1x_2} \cdots b^{1}_{x_{k-2}x_{k-1}}\, b^{2}_{x_{k-1}j}) \\ & + \cdots + f_k(b^{1}_{x_ix_j} \cdots) b^{2}_{ik_1}\, b^{2}_{x_1j} \Big]. \\ = {} & (-1)^{k+1}\, k!(B_2B_1^{k-1}B_2)_{ij} + f_1(b^{1}_{x_ix_j})(B_2B_1^{k-2}B_2)_{ij} \\ & + \cdots + f_k(b^{1}_{x_ix_j})(B_2B_2)_{ij}. \end{aligned}$$

If

$$a_{k+1} = 0,\ a_k = 0,\ \cdots\ a_2 = 0, \quad \text{then}$$

by consideration of $a_k, \cdots, a_2$,

$$(B_3 B_1^m B_2)_{ij} = 0 \qquad m = 0, 1, \cdots, k-2,$$

and

$$(B_3 B_1^{k-1} B_2)_{ij} = 0 \qquad \text{all } i \text{ and } j,$$

so

$$B_3 B_1^{k-1} B_2 = 0.$$

If

$$B_3 B_1^{k-1} B_2 \neq 0,$$

some $(B_3 B_1^{k-1} B_2)_{ij} \neq 0$, and if

$$a_2 = 0,\ \cdots,\ a_k = 0,$$

then $a_{k+1} \neq 0$, so a minor of order $r + 1$ exists which is not identically zero. Since the minor is of degree $r + 1$, $A(t)$ is of constant rank r over F' only if

$$B_3 B_1^m B_2 = 0, \qquad m = 0, 1, \cdots, r - 1,$$

as was to be shown.

Note: If $t = \frac{1}{u}$, then

$$I + B_1 t = \frac{1}{u}(Iu + B_1).$$

Let

$$f(u) \equiv | Iu + B_1| \equiv u^r + b_{r-1} u^{r-1} + \cdots + b_o,$$

then B_1 satisfies its characteristic equation,

$$f(B_1) \equiv B_1^r + b_{r-1} B_1^{r-1} + \cdots + b_o I = 0.$$

But

$$B_2 f(B_1) B_2 = B_2 B_1^r B_2 + b_{r-1} B_2 B_1^{r-1} B_2 + \cdots + b_o B_2 B_2 = 0,$$

and from Lemma IV

$$B_2 B_1^{r-1} B_2 = 0, \cdots, B_2 B_2 = 0,$$

hence

$$B_2 B_1^r B_2 = 0.$$

If

$$B_2 B_1^p B_2 = 0, \quad p = 0,1,2, \cdots, (r-1+k), \text{ then}$$

$$B_2 f(B_1) B_1^{p-r+1} B_2 = B_2 B_1^{p+1} B_2 + b_{r-1} B_2 B_1^p B_2 + \cdots + B_2 B_1^{p-r+1} B_2 = 0,$$

and

$$B_2 B_1^{p+1} B_2 = 0.$$

Thus

$$B_2B_1^{p+1}B_2 = 0, \qquad p = 0,1,2,\cdots,(r+k).$$

The relation is true for $k = 1$, hence for all k.

Further, if B_1^{-1} exists, then

$$B_1^{-1} = c_oI + c_1B +\cdots+ B_1^{r-1}$$

and

$$B_2B_1^{-1}B_2 = 0.$$

$B_1^{-2} = (B_1^{-1})(B_1^{-1})$ and similar reasoning holds.

Thus

$$B_2B_1^kB_2 = 0 \text{ for all integers for which } B_1^k \text{ exists.}$$

Making use of the following known theorem[1]: If a minor of order r in a matrix is not zero, and if every bordered determinant of this minor is zero, every $r + 1$ ordered minor of the matrix is zero; we prove the following theorem.

Theorem III. A necessary and sufficient condition for every minor of order $r + 1$ to vanish identically in a constant rank matrix of the type $A(t)$ is

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[1] Muir and Metzler. A treatise on the theory of determinants. Albany. Privately published. p. 230. 1930.

1) $B_4 = 0$,

2) $B_3 B_1^k B_2 = 0, \quad k = 0,1,2,\cdots,r-1.$

Proof. The necessity of the condition has been shown in Lemmas III and IV. Also, by those Lemmas, the conditions are sufficient for the identical vanishing of all bordered determinants of $I + B_1t$. Now suppose $|I + B_1t| = 0$. This is a non-identically zero polynomial of degree at most r in t. Thus there exist at most r distinct values of r for which $|I + B_1t| = 0$. Any minor of order r + 1 is a polynomial in t of degree at most r + 1. Deleting from consideration the at most r values of t for which $|I + B_1T| = 0$, the theorem quoted above holds. But then any minor of order r + 1 is zero for more than 2r + 1 - r = r + 1 distinct values of t, and hence identically zero. Thus the sufficiency is shown and the theorem proved.

Thus, if A(t) is of constant rank r over F', there exists a form obtainable by elementary transformations, independent of t, namely

$$PA(t)Q = \begin{pmatrix} I + B_1t & B_2t \\ B_3t & 0 \end{pmatrix},$$

such that $B_3 B_1^k B_2 = 0$, all k. This will be called the primary canonical form for a constant rank linear matrix. Theorem III is a necessary condition that any linear matrix be of constant rank.

Two Special Cases

<u>Theorem IV</u>. If, in the primary canonical form, the conditions of Theorem III are satisfied, a sufficient condition for $A(t)$ to be of constant rank over $\overline{F}$, the algebraic closure of F, is that the rank of B_2 plus the rank of B_3 be r.

Proof. If the rank of B_2 plus the rank of B_3 is r, let the rank of B_2 be k. Then there exists in B_2 a non-zero minor of order k, and in B_3 a non-zero minor of order $r-k$. The minor of order r of $PA(t)Q$ which has the k rows and columns of the non-zero minor of B_2 and the $r-k$ rows and columns of the non-zero minor of B_3 is not zero, for in the Laplace expansion of the first k rows every term in the expansion must be zero except the term which is the product of the non-zero minors of B_2 and B_3. Hence there exists a minor of order r of the form

$$Kt^r,$$

but the minor $I + B_1t = 0$ has no zero root, and Kt^r and $I + B_1t$ can have no common zero.

<u>Theorem V.</u> If in the primary canonical form $B_2 = 0$, $B_3 = 0$, a necessary and sufficient condition that $A(t)$ be of constant rank r over F' is that $|I + B_1 t|$ have no zero if F'.

Proof. $I + B_1 t$ is the only possible non-zero minor of $PA(t)Q$.

The Reduced Canonical Form

If the elementary divisors of $I + B_1 t$ are not all linear, it may be possible to make them linear by means of the elementary transformations,

$$\begin{pmatrix} I & X \\ 0 & I \end{pmatrix}, \begin{pmatrix} I & 0 \\ Y & I \end{pmatrix},$$

X and Y being completely arbitrary. Of course, the possibility of this operation will depend on the non-zero elements of B_2, B_3, and their positions. In particular, when the elementary divisors of $I + B_1 t$ are linear, there exists an elementary transformation, R, such that

$$R\, B_1 R^{-1} = D_1,$$

where

$$D_1 = \begin{pmatrix} d^1_{11} & & & \\ & d^1_{22} & & \\ & & \ddots & \\ & & & d^1_{rr} \end{pmatrix},$$

and using the elementary transformations

$$\begin{pmatrix} R & 0 \\ 0 & I \end{pmatrix}, \begin{pmatrix} R^{-1} & 0 \\ 0 & I \end{pmatrix}, \text{ we obtain}$$

$$\begin{pmatrix} R & 0 \\ 0 & I \end{pmatrix} PA(t)Q \begin{pmatrix} R^{-1} & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} I + D_1 t & D_2 t \\ D_3 t & 0 \end{pmatrix},$$

where

$$D_3 = B_3 R^{-1}, \quad D_2 = RB_2, \text{ and}$$

$$\begin{aligned} D_3 D_1^k D_2 &= B_3 R^{-1} (RB_1 R^{-1})^k R\, B_2 \\ &= B_3 R^{-1} R\, B_1^k R^{-1} R\, B_2 \\ &= B_3 B_1^k B_2 = 0. \end{aligned}$$

The matrix

$$\begin{pmatrix} I + D_1 t & D_2 t \\ D_3 t & 0 \end{pmatrix}$$

will be called the reduced canonical form for $A(t)$. When the form exists, we have

<u>Theorem VI</u>. If $A(t)$ possesses a reduced canonical form, then a set of necessary and sufficient conditions for $A(t)$ to be of constant rank r over F' is:

1) the conditions of Theorem III;
2) if d_{ii}^1 in the reduced canonical form is in F', then either the i^{th} row of D_2 or the i^{th} column of D_3 have a non-zero element.

Proof. The method of proof is to show the conditions of Lemma I are satisfied. First we see that the conditions of Theorem III are necessary and sufficient for the identical vanishing of all minors of order $r + 1$. Further, if $B_2 = 0$ and $B_3 = 0$, then Theorem V is valid, and for this special case condition 2 above is necessary and sufficient that the greatest common divisor of minors of order r has no zero in F'. On the other hand, if $D_2 \neq 0$ or $D_3 \neq 0$, the reduced canonical form has the appearance:

$$\begin{pmatrix} 1+d_{11}^1 t & & & & d_{1i}^2 t \cdots & \\ & 1+d_{22} t & & & d_{2i}^2 t \cdots & \\ & & \ddots & & & \\ & & & 1+d_{rr}^1 t & d_{ri} t \cdots & \\ d_{j1}^3 t & d_{j2} t & \cdots & d_{jr}^3 t & 0 & \\ \cdots & & & & & \ddots \end{pmatrix}.$$

If d^1_{ii} is in F', then $-\frac{1}{d^1_{ii}}$ is in F'. If a column of D_2 or row of D_1 replaces a column or row of $I + D_1t$, then the only possible term of the resulting minor is the product of the diagonal elements. But this product is zero unless the element from the column or row which lies on the diagonal is not zero. Now each replacement, in effect, will yield a new minor of order r with the factor $(1 + d^1_{ii}t)$ replaced by the factor t, if the minor is not zero. Furthermore, replacing more than one row or column will alter only the factors of $I + D_1t$ of the replaced rows or columns, and the factor t^j, say, is introduced. Thus, for the greatest common divisor of minors of order r to have no zero in F', there must exist a minor of order r in which any arbitrary set of factors $1 + d_{ii}t$, (d^1_{ii} in F'), has been deleted by using columns of D_2 and rows of D_2, thus completing the proof of the theorem.

IV. MULTILINEAR MATRICES OF CONSTANT RANK

In this chapter certain consequences for multilinear matrices of constant rank are deduced from the results of Chapter III.

Let

$$A(t_1, t_2, \cdots, t_k) = A_0 + A_1 t_1 + A_2 t_2 + \cdots + A_k t_k$$

where the matric quantities of A_i are elements of F_i, and independent of the t_i's. Let all F_i be sub-fields of a closed algebraic field $\overline{F}$, and the t_i's independent scalar variables in sub-fields F_i' of $\overline{F}$. Assume each sub-field F_i' contains more than $2r + 1$ distinct elements where r is the rank of the matrix.

Restating Def. I for this case, we have

Definition III. $A(t_1, t_2, \cdots, t_k)$ is of constant rank r over $F_1', F_2', \cdots, F_k'$ if and only if $A(t_1, t_2, \cdots, t_k)$ if exactly of rank r for every set of t_i in F_i' $i = 1, 2, \cdots, k$.

Lemma V. If $A(t_1, t_2, \cdots, t_k)$ is of constant rank r over F', A_0 is of rank r.

Proof. 0 is an element of $F_1', F_2', \cdots, F_k'$ and hence $A(0, 0, \cdots, 0) = A_0$ is of rank r.

Thus there exist constant elementary transformations P, Q such that

$$P\,A_0Q = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix},$$

and

$$PA(t_1,t_2,\cdots,t_k)Q = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} + PA_1Qt_1 + PA_2Qt_2 + \cdots + PA_kQt_k.$$

Partitioning PA_iQ arbitrarily as PA_0Q, we have

$$PA(t_1,t_2,\cdots,t_k)Q = \begin{pmatrix} I+B_1^1t_2 + B_1^2t_2 + \cdots + B_1^kt_k & B_2^1t_1 + \cdots + B_2^kt_k \\ B_3^1t_1 + \cdots + B_3^kt_k & B_4^1t_1 + \cdots + B_2^kt_k \end{pmatrix}.$$

<u>Theorem VII</u>. If $A(t_1,t_2,\cdots,t_k)$ is of constant rank r over F_1', F_2', $\cdots$, F_k', then

$$B_4^i = 0, \quad \text{and}$$

$$B_3^i(B_1)^pB_2^i = 0, \qquad \begin{matrix} i = 1,2,\cdots,k \\ p = 1,2,3,\cdots \end{matrix} \quad .$$

Proof. Let

$$t_j = 0, \qquad i \neq j.$$

Then

$$PA(0, \cdots, t_i, \cdots, 0)Q = \begin{pmatrix} I + B_1^i t_i & B_2^i t_i \\ B_3^i t_i & B_4^i t_i \end{pmatrix}$$

is of constant rank r over F_i'. But by Theorem III,

$$B_4^i = 0, \text{ and}$$

$$B_3^i (B_1^i)^p B_2^i = 0, \qquad p = 1, 2, \cdots, \text{ hence}$$

letting $i = 1, 2, \cdots, k$ successively, the theorem is shown.

The conditions of this theorem are necessary conditions for a multilinear matrix to be of constant rank. The primary canonical form for the multilinear matrix of constant rank is, then

$$\begin{pmatrix} I + B_1^1 t_1 + B_1^2 t_2 + \cdots + B_1^k t_k & B_2^1 t_1 + B_2^2 t_2 + \cdots + B_2^k t_k \\ B_3^1 t_1 + B_3^2 t_2 + \cdots + B_3^k t_k & 0 \end{pmatrix}.$$

V. APPENDIX: EXAMPLES

In this appendix we give two illustrations, first of a constant rank linear matrix, and second of a non-constant rank linear matrix.

Let

$$A(t) = \begin{pmatrix} 2t-2 & -2t-1 & -12t+1 & 6t-1 \\ -3t+3 & 10t+1 & 39t-3 & -16t+2 \\ t-1 & -t & -6t+2 & 3t-1 \end{pmatrix}.$$

Then

$$A_0 = \begin{pmatrix} -2 & -1 & 1 & -1 \\ 3 & 1 & -3 & 2 \\ -1 & 0 & 2 & -1 \end{pmatrix}, \text{ and}$$

$$A_1 = \begin{pmatrix} 2 & -2 & -12 & 6 \\ -3 & 10 & 39 & -16 \\ 1 & -1 & -6 & 3 \end{pmatrix}.$$

If

$$P = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 1 & 1 & 1 \end{pmatrix} \text{ and } Q = \begin{pmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -3 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \text{ then}$$

$$PA_0Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence, if $A(t)$ is of constant rank r over some field, then $r = 2$. Now

$$PA_1Q = \begin{pmatrix} -1 & 4 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \text{ and partitioning}$$

$$PA(t)Q = \begin{pmatrix} I + \begin{pmatrix} -1 & 4 \\ 0 & 1 \end{pmatrix} t & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} t \\ (0 \quad 1)t & (0 \quad 0)t \end{pmatrix}, \text{ we see that}$$

$B_1 = \begin{pmatrix} -1 & 4 \\ 0 & 1 \end{pmatrix}$, $B_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B_3 = (0 \quad 1)$, $B_4 = (0 \quad 0)$,

$B_1^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, so $B_1^{2p} = I$, $B_1^{2p+1} = B_1$ for all p, and

$B_3IB_2 = 0$, $B_3B_1B_2 = 0$.

Therefore, $B_3B_1^kB_2 = 0$ all k, and by the form above, $B_4 = 0$. Hence, in the primary canonical form, the necessary conditions are satisfied.

If

$$R = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad R^{-1} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}, \text{ then}$$

$$\begin{pmatrix} R & 0 \\ 0 & I \end{pmatrix} PA_1Q \begin{pmatrix} R^{-1} & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

and

$$\begin{pmatrix} R & 0 \\ 0 & I \end{pmatrix} PA(t)Q \begin{pmatrix} R^{-1} & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} 1-t & 0 & t & 0 \\ 0 & 1+t & 0 & 0 \\ 0 & t & 0 & 0 \end{pmatrix}.$$

So the reduced canonical form exists. The rank of $A(t)$ is 2, a constant, over any field, by inspection.

2. Let

$$A(t) = \begin{pmatrix} t+1 & 3t+3 & 6t+4 \\ 6t & 1 & 3t+2 \\ t+1 & 3t+3 & 4t+2 \end{pmatrix}, \text{ then}$$

$$A_0 = \begin{pmatrix} 1 & 3 & 4 \\ 0 & 1 & 2 \\ 1 & 3 & 2 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 1 & 3 & 6 \\ 6 & 0 & 3 \\ 1 & 3 & 4 \end{pmatrix}.$$

If

$$P = \begin{pmatrix} -2 & 1 & 3 \\ 3 & -2 & 3 \\ -1 & 1 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & -2 & 4 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}, \text{ then}$$

$$PA_0Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad PA_1Q = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then

$$B_1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad B_3 = (0 \quad 0), \quad B_4 = (1).$$

Hence $A(t)$ is not of constant rank for $B_4 \neq 0$.

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